MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 2

- Linear Transformations.
- The Matrix of a Linear Transformation.
- Matrix Operations.
- Inverse of a Matrix.
- Invertible Matrix Theorem.
- Inverse of a Linear Transformation.

Linear Transformations

A transformation (or mapping) T is linear if:

i) $T(u+v) = T(u) + T(v)$ for all the vectors u, v in the domain of T.

ii) $T(cu) = cT(u)$ for all the vectors u in the domain of T and all the scalars c.

Example: Let $T{:}R^2 \to R^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ x+y \end{bmatrix}.$$

Let us show that T is a linear transformation:
(i)

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right) = \begin{bmatrix} y+z \\ x+w \\ x+w+y+z \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ x+y \end{bmatrix} + \begin{bmatrix} z \\ w \\ z+w \end{bmatrix} = \begin{bmatrix} y+z \\ x+w \\ x+y+z+w \end{bmatrix}.$$

ii)

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx \\ cx+cy \end{bmatrix}$$

$$cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c\begin{bmatrix} y \\ x \\ x+y \end{bmatrix} = \begin{bmatrix} cy \\ cx \\ cx+cy \end{bmatrix}.$$

By i) and ii) T is a linear transformation.

Remark 1: Conditions for a linear transformation can be written as only one equation:

$T(cu + dv) = c\,T(u) + d\,T(v)$

for all $u,\ v \in R^n$, and for all scalars c and d.

Remark 2: For any linear transformation T,

$T(0) = T(0u) = 0T(u) = 0.$

Example: Let $T{:}R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+1 \end{bmatrix}.$$

Show that T is not a linear transformation.
Solution:

$$\begin{aligned} T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx+1 \end{bmatrix}, \\ cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= c\begin{bmatrix} y \\ x+1 \end{bmatrix} = \begin{bmatrix} cy \\ c(x+1) \end{bmatrix} = \begin{bmatrix} cy \\ cx+c \end{bmatrix}. \\ T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) &\neq cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \text{ if } c \neq 1. \end{aligned}$$

Example: For any $m \times n$ matrix A, the matrix transformation $T : R^n \longrightarrow R^m$ given by

$$T(X) = AX$$

is a linear transformation.

Solution: We need to show that if A is an $m \times n$ matrix, u and v are vectors in R^n, and c is a scalar, then:

$$\begin{aligned} A(u+v) &= Au + Av \\ A(cu) &= c(Au). \end{aligned}$$

For simplicity, take $m = 3$ and $n = 2$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$\begin{aligned} A(u+v) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11}(u_1+v_1) + a_{12}(u_2+v_2) \\ a_{21}(u_1+v_1) + a_{22}(u_2+v_2) \\ a_{31}(u_1+v_1) + a_{32}(u_2+v_2) \end{bmatrix} \\ &= \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \\ a_{31}u_1 + a_{32}u_2 \end{bmatrix} + \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \\ a_{31}v_1 + a_{32}v_2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$$

$$= Au + Av.$$

$$\begin{aligned} A(cu) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} a_{11}cu_1 + a_{12}cu_2 \\ a_{21}cu_1 + a_{22}cu_2 \\ a_{31}cu_1 + a_{32}cu_2 \end{bmatrix} \\ &= c \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \\ a_{31}u_1 + a_{32}u_2 \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = c(Au) \end{aligned}$$

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix}, \ u = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Compute Au, Av, $A(u+v)$, and $A(5u)$.
Solution:

$$Au = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+8-9 \\ 4-20-18 \\ -9+32-21 \end{bmatrix} = \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix}.$$

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2+6 \\ 0-5+12 \\ 0+8+14 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 22 \end{bmatrix}.$$

$$\begin{aligned} A(u+v) &= A(u) + A(v) \\ &= \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \\ 22 \end{bmatrix} = \begin{bmatrix} 8 \\ -27 \\ 24 \end{bmatrix}. \end{aligned}$$

$$A(5u) = 5(Au) = 5 \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -170 \\ 10 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let $T : R^2 \longrightarrow R^3$ defined by $T(X) = AX$.

a) Find $T(u)$.

b) Find a vector X in R^2 such that $T(X) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

c) Is there another vector $Y \neq X$ such that $T(Y) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

d) Is $W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ in the range of T?

Solution: a)

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$

b) We need to find a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

Corresponding augmented matrix:

$$\left[\begin{array}{rr|r} 1 & 2 & -1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

Then, $x_2 = -2$ and $x_1 = -1 - 2x_2 = 3$. Thus, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

c) Since the system

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

has a unique solution, the vector $X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is the only vector such that

$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

d) We need to check whether the equation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

has a solution:

$$\left[\begin{array}{rr|r} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right],$$

which is inconsistent.

$$\text{So, } W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ is not in the range of } T.$$

The matrix of a Linear Transformation

We have seen that for any $m \times n$ matrix A, the transformation $T : R^n \longrightarrow R^m$ defined by $T(X) = AX$ is a linear transformation.

Conversely, for any linear transformation $T : R^n \longrightarrow R^m$ there is a unique $m \times n$ matrix A such that $T(X) = AX$.
The matrix A is called the **standard matrix** for T.

Examples:

$$T : R^2 \longrightarrow R^2, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{A_{2\times 2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T : R^2 \longrightarrow R^3, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ -x \\ 2x+y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}}_{A_{3\times 2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T : R^3 \longrightarrow R^2, \quad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-3z \\ 4x-y+5z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 5 \end{bmatrix}}_{A_{2\times 3}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition: Standard basis for R^n is given by

$$\mathcal{B} = \{e_1, e_2, ..., e_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

Let $T : R^n \longrightarrow R^m$ be a linear transformation such that

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, ..., T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix A given by

$$A_T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

is the **standard matrix** for T.

Example: Find the standard matrix of the linear transformation $T : R^3 \longrightarrow R^4$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ 3x - 4y + 5z \\ y + z \\ -3x + 5y - 4z \end{bmatrix}.$$

Solution:

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix},$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ 1 \\ -4 \end{bmatrix}.$$

So, the standard matrix of T is

$$A_T = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}. \text{ Then, } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example: Reflection in the $x-$ axis is a linear transformation:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Example: Let $T : R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

T rotates the vectors in R^2 90° counterclockwise about the origin.

Examples (Rotations): A rotation in the plane is a linear transformation:
Let $T : R^2 \to R^2$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ be any point in R^2. Then,

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} + y\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation matrix}}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The linear transformation $R_\varphi : R^2 \to R^2$

$$R_\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

rotates the vectors counterclockwise by angle φ.

For $\varphi = \pi/2$: $R_{\frac{\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$

For $\varphi = \pi/4$: $R_{\frac{\pi}{4}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} x-y \\ x+y \end{bmatrix}.$

For $\varphi = \pi/3$: $R_{\frac{\pi}{3}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}.$

For $\varphi = \pi$: $R_\pi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$

For $\varphi = 2\pi$: $R_{2\pi}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$

Horizontal shear and Vertical shear: The linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix}$$

is called a **horizontal shear** (or $x-$shear).
Similarly, the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ax+y \end{bmatrix}$$

is called a **vertical shear** (or $y-$shear) for any number a.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

The transformation $T : R^2 \longrightarrow R^2$ given by $T(X) = AX$ is a positive $x-$shear.

Find $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$.
Solution:

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

The linear transformation $T : R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix}$$

is called a **contraction** if $0 \le r \le 1$ and a **dilation** if $r > 1$.

Definition: Let $T : R^n \longrightarrow R^m$ be a linear transformation.

(1) If for each vector b in R^m there is at least one vector X in R^n such that $T(X) = b$, then T is said to be **onto** R^m.

(2) If each vector b in R^m is the image of at most one vector X in R^n, then T is said to be **one-to-one**.

Example: Let $T : R^3 \longrightarrow R^2$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\4\end{bmatrix},\quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\end{bmatrix}.$$

a) Is T onto? Explain your answer.

b) Is T one-to-one? Explain your answer.

Solution:

$$A=\begin{bmatrix} T(e_1) & T(e_2) & T(e_3)\end{bmatrix}=\begin{bmatrix}1&2&3\\2&4&6\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&0&0\end{bmatrix}.$$

a) Since A does not have a pivot position in each row, T is not onto.

b) $AX=0$ has infinitely many solutions. So, T is not one-to-one.

Theorem: Let $T : R^n \longrightarrow R^m$ be a linear transformation, and let $A_{m\times n}$ be the standard matrix of T.

(1) T is one-to-one $\Longleftrightarrow$ $T(X)=0$ has only the trivial solution.

$\Longleftrightarrow$ Columns of A are linearly independent.

$\Longleftrightarrow$ Each column of A has a pivot position.

(2) T is onto $\Longleftrightarrow$ Columns of A span R^m.

$\Longleftrightarrow$ Each row of A has a pivot position.

$\Longleftrightarrow$ For each $b \in R^m$, $AX=b$ is consistent.

Example: $T : R^3 \longrightarrow R^4$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ x_1 - x_2 \\ x_2 + x_3 \\ x_1 - x_2 \end{bmatrix}.$$

i) Is T one-to-one? Explain your answer.

ii) Is T onto? Explain your answer.

Solution:

$$A_T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

i) Since each column of REF of A_T has a pivot position, the columns of A_T are linearly independent, and so T is one-to-one.

ii) Since each row of REF of A_T does not have a pivot position, T is not onto.

Remark: If $T : R^n \longrightarrow R^m$ is a linear transformation and $\{v_1, v_2, v_3\}$ is a linearly dependent set in R^n, then $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent in R^m.

Example: The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \ v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

are linearly dependent since $v_3 = -2v_1 + v_2$.
For any linear transformation $T : R^3 \longrightarrow R^m$

$$T(v_3) = T(-2v_1 + v_2) = -2T(v_1) + T(v_2).$$

Thus, the vectors

$$T(v_1),\ T(v_2),\ T(v_3)$$

are also linearly dependent.

Remark: If $\{v_1, v_2, v_3\}$ is a linearly independent set, then $\{T(v_1), T(v_2), T(v_3)\}$ does not need to be linearly independent.

Example: Consider the linear transformation $T : R^3 \longrightarrow R^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 4x_2 + 6x_3 \end{bmatrix}.$$

The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent but

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, T(e_3) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

are linearly dependent.

MATRIX OPERATIONS

An $m \times n$ matrix A is a rectangular array of numbers with m rows and n columns. We denote an $m \times n$ matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij}) \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$$

The entry a_{ij} is called the (i, j)-entry of A.

If $m = n$, then A is called a **square matrix of order n**, and the entries a_{11}, a_{22},..., a_{nn} form the **main diagonal** of A.

The **identity matrix** I_n of size $n \times n$ is a matrix such that all the main diagonal entries are 1, and all other entries are 0.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the identity matrices of sizes 2×2 and 3×3, respectively.
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not an identity matrix.

An $m \times n$ matrix whose entries are all zero is a **<u>zero matrix</u>**.

Examples: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Let $A_{m\times n} = (a_{ij})$ and $B_{r\times s} = (b_{ij})$. Then,

$$A = B \iff \begin{cases} m = r \text{ and } n = s \text{ and} \\ a_{ij} = b_{ij} \text{ for all } i \text{ and } j. \end{cases}$$

Example: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -4 \\ 5 & 6 & 8 \end{bmatrix}$. Then,

$$A = B \iff \begin{cases} a_{11} = 2, a_{12} = 1, a_{13} = -4, \\ a_{21} = 5, a_{22} = 6, a_{23} = 8. \end{cases}$$

Example: The matrices $R_{1\times 3} = \begin{bmatrix} 5 & -2 & 4 \end{bmatrix}$ and $C_{3\times 1} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$ have the same entries but they are not equal (they have different sizes).

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3}, \quad B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 8 & 6 \end{bmatrix}_{3\times 2}, \text{ and } C = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & 1 \end{bmatrix}_{2\times 3}.$$

Then

$$A + C = \begin{bmatrix} 1+3 & 2+5 & 3+0 \\ 4-1 & 5+2 & 7+1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 3 \\ 3 & 7 & 8 \end{bmatrix}_{2\times 3},$$

$$A + B = \text{ can not be done.}$$

If A and B are matrices of the same size, their sum $A + B$ is a matrix of the same size formed by adding corresponding entries in A and B.

If A is a matrix and c is a scalar, the product cA is a matrix of the same size as A in which every entry is multiplied by c.

Example:

$$5\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 10 & 15 \end{bmatrix}.$$

Theorem: Let A, B and C be matrices of the same size, and let r and s be scalars. Then

$$\begin{aligned} & A + B = B + A \\ & (A + B) + C = A + (B + C) \\ & A + 0 = A \\ & r(A + B) = rA + rB \\ & (r + s)A = rA + sA \\ & r(sA) = (rs)A \end{aligned}$$

Matrix Multiplication: $\boxed{A_{m\times n}B_{n\times k} = C_{m\times k}}$

Example:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 1 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 4+2+24 & -1+6+18 \\ 16+5+56 & -4+15+42 \end{bmatrix} = \begin{bmatrix} 30 & 23 \\ 77 & 53 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 4 & -1 \\ 1 & 3 \\ 8 & 6 \end{bmatrix}_{3\times 2}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 0 & 3 & 5 \\ 13 & 17 & 24 \\ 32 & 46 & 66 \end{bmatrix}_{3\times 3}$$

Matrix product is not commutative.

$$A_{2\times 3}C_{2\times 3} = \text{ can not be done.}$$
$$C_{2\times 3}A_{2\times 3} = \text{ can not be done.}$$

Example: Let

$$A = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, $AB = 0$ but $A \neq 0$, $B \neq 0$.

Example: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

$$\left.\begin{aligned} AB &= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \\ AC &= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \end{aligned}\right\} \begin{matrix} AB = AC \\ B \neq C \end{matrix}$$

Warnings:
- In general, $AB \neq BA$.
- If $AB = AC$, then it is not true in general that $B = C$.
- If $AB = 0$ (zero matrix), you cannot conclude in general that either $A = 0$ or $B = 0$.

Theorem: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$$r(AB) = (rA)B = A(rB) \text{ for any scalar } r$$
$$I_m A = A = A I_n$$

Example: Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Find A^2, A^3 and A^4.

Solution:

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly,

$$A^4 = AA^3 = \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}$.

For what value(s) of k, $AB = BA$?

Solution:

$$AB = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} = \begin{bmatrix} 1 & 12-4k \\ -30 & -20+k \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -24 \\ 15-5k & -20+k \end{bmatrix}$$

Then $AB = BA \Leftrightarrow \left\{ \begin{array}{l} 15 - 5k = -30 \text{ and} \\ 12 - 4k = -24. \end{array} \right\}$

Thus, $AB = BA \Leftrightarrow k = 9$.

Example: Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \text{ and } AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}.$$

Determine the first and second columns of B.

Solution: Let $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}.$$

This gives that

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} a - 2d \\ -2a + 5d \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]. \Longrightarrow \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

To determine the second column of B is left as an exercise.

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix whose columns are the corresponding rows of A, and denoted by A^T.

Example:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & 7 & 8 \\ 0 & 1 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & 6 & 0 \\ -1 & 7 & 1 \\ 2 & 8 & 5 \end{bmatrix}.$$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Properties of Transpose:

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(rA)^T = rA^T$$

$$(AB)^T = B^T A^T.$$

Example: Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}. \text{ Then,}$$

$$(AB)^T = B^T A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 14 & 28 \\ 15 & 35 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1 & 0 & b \\ a & -1 & 0 \\ 0 & b & 1 \end{bmatrix}$.

For what values of a and b:

(i) $A^2 = I_3$? **(ii)** $AA^T = I_3$?

Solution:

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 & b \\ a & -1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ a & -1 & 0 \\ 0 & b & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & b^2 & 2b \\ 0 & 1 & ab \\ ab & 0 & 1 \end{bmatrix} \\ &= I_3 \iff a \in R \text{ and } b = 0. \end{aligned}$$

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 0 & b \\ a & -1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & -1 & b \\ b & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+b^2 & a & b \\ a & 1+a^2 & -b \\ b & -b & 1+b^2 \end{bmatrix} \\ &= I_3 \iff a = 0 \text{ and } b = 0. \end{aligned}$$

Inverse of a Matrix

Definition: Let A be an $\underline{n \times n \text{ square}}$ matrix. If there is a matrix B such that $AB = BA = I_n$
then A is said to be **invertible** and B is called the inverse of A.

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A is invertible $\Leftrightarrow ad - bc \neq 0$.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.
The **determinant** of A is $\det A = ad - bc$.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Since $\det A = 4 - 6 = -2 \neq 0$, A is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \text{ and } AA^{-1} = A^{-1}A = I_2.$$

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.
Since $\det A = 0$, A is not invertible.

Note: Although A is not a zero matrix, it is not invertible.

Theorem: If A is an invertible $n \times n$ matrix, then for each $b \in R^n$, the equation $AX = b$ has the unique solution

$$X = A^{-1}b.$$

Example: Solve the equation

$$AX = \begin{bmatrix} -5 & -11 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution:

$$\det A = -35 - (-33) = -2 \neq 0.$$

Thus, A is invertible, and

$$\begin{aligned} X &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & -11 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (-1/2) \begin{bmatrix} 7 & 11 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}. \end{aligned}$$

Theorem: Let A and B be $n \times n$ invertible matrices and k be a <u>non-zero</u> scalar. Then,

$$(A^{-1})^{-1} = A$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$
$$(kA)^{-1} = (1/k)A^{-1}.$$

Example:
Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$.

Find $(AB)^{-1}$, $(5A)^{-1}$, $(B^T)^{-1}$, and $((AB)^T)^{-1}$.

Solution:

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{3} \begin{bmatrix} 4 & 1 \\ 2 & -4 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} (5A)^{-1} &= \frac{1}{5}A^{-1} = \frac{1}{5} \left(\frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1/15 & -2/5 \\ 1/15 & 1/15 \end{bmatrix}. \end{aligned}$$

$$(B^T)^{-1} = (B^{-1})^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

$$\begin{aligned} ((AB)^T)^{-1} &= ((AB)^{-1})^T = \frac{1}{3} \begin{bmatrix} 4 & 1 \\ 2 & -4 \end{bmatrix}^T \\ &= \frac{1}{3} \begin{bmatrix} 4 & 2 \\ 1 & -4 \end{bmatrix}. \end{aligned}$$

Exercise: Let

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Verify that $CA = I_2$. Is A invertible? Explain your answer.

Example: Find A when $(A^T - 2I)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

Solution:

$$\begin{aligned}
(A^T - 2I)^{-1} &= 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}. \\
(A^T - 2I) &= \left(2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \\
&= \frac{1}{2}\left(\frac{1}{3-2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A^T &= 2I + \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \\
&= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 7/2 & -1/2 \\ -1 & 5/2 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 7/2 & -1 \\ -1/2 & 5/2 \end{bmatrix}.
\end{aligned}$$

Elementary Matrices

An elementary matrix E is a square matrix obtained by performing a single row operation on an identity matrix.

Examples: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(I) $R_2 \longleftrightarrow R_3 : \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1$

(II) $R_1' = 4R_1 : \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$

(III) $R_2' = \underbrace{R_2 - 7R_1}_{a_{21}=-7} \quad \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \end{bmatrix},$$

$$E_2A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 4a & 4b \\ c & d \\ e & f \end{bmatrix},$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ c-7a & d-7b \\ e & f \end{bmatrix}.$$

Note that the matrices E_1A, E_2A and E_3A are the matrices obtained from A by performing the elementary row operations I, II and III, respectively.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Elementary Row Operation	Corresponding Inverse Operation
$R_i \leftrightarrow R_j$	$R_j \leftrightarrow R_i$
$R_i' = cR_i$	$R_i' = (1/c)R_i$
$R_i' = R_i + cR_j$	$R_i' = R_i - cR_j$

Example:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1.$$

$$E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow E_2^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Perform the following row operations on I_2, and write the corresponding elementary matrices and their inverses.

- $R_1' = \underbrace{R_1 - 3R_2}_{a_{12}=-3 \text{ in } E_1}$:

$$E_1 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \Longrightarrow E_1^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

- $R_2' = 4R_2$:

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Longrightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

- $R_1 \leftrightarrow R_2$:

$$E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Longrightarrow E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example: Write $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and A^{-1}

as products of elementary matrices.

Solution: First we compute A^{-1}.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \quad {R_2}' = R_2 - R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad {R_1}' = R_1 - R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

- $R_2{}' = R_2 - R_1 : \implies E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

- $R_1{}' = R_1 - R_2 : \implies E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Therefore,

$$A^{-1} = E_2E_1 \text{ and } A = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1}$$

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

a) Find the inverse of A.
b) Write A and A^{-1} as products of elementary matrices.
Solution: a)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3' = R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \quad R_2{}' = \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

b) First we need to write the corresponding elementary matrix for each row operation.

- $R_3' = \underbrace{R_3 - 4R_1}_{a_{31}=-4} \Longrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$

- $R_2 \longleftrightarrow R_3 \Longrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

- ${R_2}' = \frac{1}{2}R_2 \Longrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Note that $E_3E_2E_1A = I$, and so

$$A^{-1} = E_3E_2E_1 \text{ and } A = E_1^{-1}E_2^{-1}E_3^{-1}.$$

Example: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$

Find the inverse of A.
Solution:

$$\left[\begin{array}{rrr|rrr} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_2' = R_2 - 2R_1$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & -2 \end{array}\right] \quad R_3' = R_3 - 2R_2$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & -2 \end{array}\right] \quad R_3' = -\frac{1}{3}R_3$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 & -1/3 & 2/3 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_3 \\ R_1' = R_1 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 2/3 \end{array}\right] \quad R_1^{'} = R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 2/3 \end{array}\right].$$

Thus,

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & -2 \\ 2 & -1 & 2 \end{bmatrix}.$$

The Invertible Matrix Theorem

Let A be a square nxn matrix. Then the following statements are equivalent.

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. REF of A has a pivot position in each row.

d. REF of A has a pivot position in each column.

e. The equation $Ax = 0$ has only the trivial solution.

f. The columns of A form a linearly independent set.

g. The equation $Ax = b$ is consistent for every b in R^n.

h. The columns of A span R^n.

i. A^T is an invertible matrix.

j. There is an nxn matrix C such that
$AC = CA = I$.

k. The linear transformation $T(x) = Ax$ is one-to-one.

l. The linear transformation $T(x) = Ax$ is onto.

Note that if the matrix A is not a square matrix then the Invertible Matrix Theorem does not apply.

Example: Consider the following matrix A.

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & -4 & 7 \\ -1 & 5 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

• Since row echelon form of A has a pivot position in each column, columns of A are linearly independent.
• Since each row of REF of A does not have a pivot, columns of A do not span R^4.
• Columns of A are linearly independent , but they do not span R^4.
• Since A is not a square matrix, the invertible matrix theorem does not apply.

Example: By Invertible matrix theorem, the matrix

$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 9 \end{bmatrix}$ is invertible since A is a 3×3 matrix, and it has 3 pivot positions.

Invertible Linear Transformations

Let $T : R^n \longrightarrow R^n$ be a linear transformation. T is said to be invertible if there exists a linear transformation $S : R^n \longrightarrow R^n$ such that

$$S(T(X)) = X \text{ and } T(S(X)) = X.$$

for all X in R^n.

Theorem: Let $T : R^n \longrightarrow R^n$ be a linear transformation, and let A be the standard matrix for T. Then, T is invertible if and only if A is invertible, and the inverse of T is given by $T^{-1}(X) = A^{-1}X$.

Example: Let $T : R^2 \longrightarrow R^2$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 6x_1 - 8x_2 \\ -5x_1 + 7x_2 \end{bmatrix}.$$

Show that T is invertible and find a formula for T^{-1}.
Solution: The standard matrix of T is

$$A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}.$$

$$\det A = 42 - 40 = 2 \neq 0.$$

A is invertible, and so T is invertible.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix}, \text{ thus}$$

$$\begin{aligned} T^{-1}(X) = A^{-1}X &= \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} (7/2)x_1 + 4x_2 \\ (5/2)x_1 + 3x_2 \end{bmatrix}. \end{aligned}$$

Let us verify that $T(T^{-1}(X)) = T^{-1}(T(X)) = X$:

$$\begin{aligned} T(T^{-1}(X)) &= T(T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} \left(\begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X. \end{aligned}$$

Similarly, $T^{-1}(T(X)) = X$ as well.